DETERMINATION OF VARIABLE COEFFICIENT OF THERMAL DIFFUSIVITY

Yu. I. Babenko

UDC 536.24.02

The coefficient of thermal diffusivity as a function of time is determined for a given temperature and its gradient on the boundary of a semiinfinite region.

The problem

$$\left[\frac{\partial}{\partial t}-a(t)\frac{\partial^2}{\partial x^2}\right]T=0,\quad 0\leqslant x<\infty,\quad 0<t<\infty,$$
$$T|_{x=0}=T_0(t);\quad (\partial T/\partial x)_{x=0}=q_0(t);\ T|_{x=\infty}=0;\ T|_{t=0}=0. \tag{1}$$

is considered. It is required to find the variable coefficient $a(t)$ of thermal diffusivity by using the "superfluous" boundary condition.

First, the solution is stated for the direct problem [1]. The substitution of

$$\tau=\int_0^t a(t)\,dt,\quad a=d\tau/dt \tag{2}$$

simplifies Eq. (1) to

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)T=0.$$

Now one can easily obtain the expression linking the temperature to its gradient on the boundary of the region [2, 3], namely,

$$-q_0(\tau)=\frac{d^{1/2}}{d\tau^{1/2}}T_0(\tau),\quad -T_0(\tau)=\frac{d^{-1/2}}{d\tau^{-1/2}}q_0(\tau). \tag{3}$$

The partial differentiation operators are given by the expressions

$$\frac{d^\nu f(t)}{dt^\nu}=\frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t f(z)(t-z)^{1-\nu}dz,\quad -\infty<\nu<1, \tag{4}$$

$$\frac{d^\nu t^\mu}{dt^\nu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}t^{\mu-\nu}. \tag{5}$$

In the above f(t) denotes a function with a finite number of discontinuities of the first kind.

Returning to the t variable the integral equations which determine $a(t)$ can be obtained

---

Institute of Applied Chemistry, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 2, pp. 341-344, August, 1975. Original article submitted October 14, 1974.

from (3) provided t_0 and q_0 are known:

$$-q_0(t) = \pi^{-1/2}[a(t)]^{-1}\frac{d}{dt}\int_0^{\tau(t)} T_0[\tau(z)][\tau(t)-z]^{-1/2}dz, \quad (6)$$

$$-T_0(t) = \pi^{-1/2}\int_0^{\tau(t)} q_0[\tau(z)][\tau(t)-z]^{-1/2}dz. \quad (7)$$

By $T_0[\tau(z)]$ and $q_0[\tau(z)]$ one understands that T_0 and q_0 are expressed as functions of τ and then τ being replaced by z.

By using the definition (4) it can be shown that the solutions (6) and (7) only exist if $\lim_{t\to 0} T_0,\ q_0 = o(t^{-1})$. A trivial example is as follows: $T_0 = 1$, $q_0 = (\pi t)^{-1/2}$, $a = 1$.

In [4] an equation similar to (6) was obtained in the case of $T_0(0) = 0$. The function $a(t)$ was obtained numerically in [5]. An analytic solution of (1) will now be given by us.

From (6) and (7), with (5) and (2) taken into account, one obtains at once the solutions for the two cases which are of importance in practice; namely, if $T_0 = \alpha_0 = \text{const}$, $q_0 = q_0(t)$ or $q_0 = \beta_0 = \text{const}$, $T_0 = T_0(t)$:

$$a(t) = -2\pi^{-1}\alpha_0^{-2}q_0^{-3}(dq_0/dt), \quad (8)$$

$$a(t) = \frac{1}{2}\pi\beta_0^{-2}T_0(dT_0/dt). \quad (9)$$

Let us now consider the case in which T_0 and q_0 are given as power series,

$$T_0 = \sum_{n=0}^{\infty}\alpha_n t^{n+\mu}, \quad (10)$$

$$q_0 = \sum_{n=0}^{\infty}\beta_n t^{n+\mu-\frac{1}{2}}. \quad (11)$$

Then $a(t)$ can also be found as a power series,

$$a = \sum_{n=0}^{\infty}\gamma_n t^n. \quad (12)$$

One can now substitute the series (10)-(12) into (6) or (7); then by using (5) and comparing the coefficients of equal powers of t one obtains a system of algebraic equations which can be used to determine all γ_n. These calculations are omitted here, since a more convenient way of finding $a(t)$ is given below.

It can be seen from (6) that if the series (10) and (12) have at least one finite radius of convergence, then the series (11) has also a finite radius of convergence. Therefore, there is a countless number of cases in which the convergent series (10) and (11) possess an associated series (12) also with a finite radius of convergence. However, we do not know whether this is valid for all convergent series (10) and (11).

Remark. The more general case can be considered in a similar way in which T_0 and q_0 are given as sums of several series with different μ. The solution is again sought in the form (12), the algebraic systems which determine the contribution to γ_n of each series being independent of each other.

For the problem (1) the relation between T_0 and q_0 can be written down with the aid of formulas in [2] as

$$-\frac{q_0(t)}{k(t)}=\Big[\frac{d^{1/2}}{dt^{1/2}}-\frac{1}{4}\cdot\frac{k'}{k}\cdot\frac{d^{-1/2}}{dt^{-1/2}}+\left(\frac{1}{8}\cdot\frac{k''}{k}-\frac{3}{32}\cdot\frac{k'^2}{k^2}\right)\frac{d^{-3/2}}{dt^{-3/2}}+$$

$$+\left(-\frac{5}{64}\cdot\frac{k'''}{k}+\frac{15}{64}\cdot\frac{k'k''}{k^2}-\frac{15}{128}\cdot\frac{k'^3}{k^3}\right)\frac{d^{-5/2}}{dt^{-5/2}}+$$

$$+\left(\frac{7}{128}\cdot\frac{k^{\mathrm{IV}}}{k}-\frac{63}{256}\,\frac{k'k^{\mathrm{III}}}{k^2}-\frac{7}{32}\,\frac{k''^2}{k^2}+\right.$$

$$\left.+\frac{21}{32}\cdot\frac{k''k'^2}{k^3}-\frac{425}{2048}\cdot\frac{k'^4}{k^4}\right)\frac{d^{-7/2}}{dt^{-7/2}}+\ldots\Big]T_0(t). \tag{13}$$

In the above the substitution

$$k(t)=[a(t)]^{-1/2}, \tag{14}$$

was used; its merit becomes apparent in deriving (13). The prime denotes differentiation with respect to time.

With T_0 and q_0 being given by the series (10) and (11), k(t) is sought also in the series form

$$k=\sum_{n=0}^{\infty}\varkappa_n t^n. \tag{15}$$

Substituting (10), (11), and (15) into (13), using (5), and by comparing the coefficients of equal powers of t, one finds for $\varkappa_n$ in the case of $\mu = 0$ the expressions

$$\pi^{-1/2}\varkappa_0=-\alpha_0^{-1}\beta_0,\quad \pi^{-1/2}\varkappa_1=4\alpha_0^{-2}\alpha_1\beta_0-2\alpha_0^{-1}\beta_1,$$

$$\pi^{-1/2}\varkappa_2=8\alpha_0^{-2}\alpha_2\beta_0-3\alpha_0^{-1}\beta_2-26\alpha_0^{-3}\alpha_1^2\beta_0+16\alpha_0^{-2}\alpha_1\beta_1-\frac{3}{2}\,\alpha_0^{-1}\beta_0^{-1}\beta_1^2,$$

$$\pi^{-1/2}\varkappa_3=\frac{192}{15}\,\alpha_0^{-2}\alpha_3\beta_0-\frac{352}{3}\,\alpha_0^{-3}\alpha_1\alpha_2\beta_0+\frac{528}{15}\,\alpha_0^{-2}\alpha_2\beta_1-$$

$$-104\alpha_0^{-4}\alpha_1^3\beta_0-\frac{148}{3}\,\alpha_0^{-3}\alpha_1^2\beta_1+\frac{62}{3}\,\alpha_0^{-2}\alpha_1\beta_0^{-1}\beta_1^2+$$

$$+\frac{228}{5}\,\alpha_0^{-2}\alpha_1\beta_2-6\alpha_0^{-1}\beta_0^{-1}\beta_1\beta_2-\alpha_0^{-1}\beta_0^{-2}\beta_1^3-4\alpha_0^{-1}\beta_3 \tag{16}$$

. . . .

The solution is given by the formulas (14)-(16).

Example. $T_0 = 1 + 0.2t$; $-(\pi t)^{1/2} q_0 = 1 + 0.2t$, $k = 1-0.40t + 0.46t^2 + 1.06t^3 + \ldots$, $a = 1 + 0.80t-0.44t^2-2.95t^3-\ldots$

It follows from (15) and (16) that if α_0 or β_0 vanish, then there is no solution. Therefore, for small α_0 or β_0 the solution is not satisfactory. In the remaining cases the solution (14)-(16) depends continuously on T_0 and q_0 for sufficiently small values of t.

The advantage of the proposed method compared to the numerical method of [5] lies in the ease of computations for sufficiently small t; its disadvantage lies in that it cannot be used if the experimental data are poorly described by a polynomial and also for large t except for the cases (8) and (9).

NOTATION

a, coefficient of thermal diffusivity; x, coordinate; t, time; T, temperature; T_0, temperature at the boundary; q_0, temperature gradient near the boundary; z, integration variable; α, β, γ, ϰ, constants of series; k, τ, auxiliary variables; μ, ν, power and differentiation indices; n, summation index.

LITERATURE CITED

1. J. Crank, The Mathematics of Diffusion, Oxford University Press (1956), p. 147.
2. Yu. I.Babenko, in: Proceedings of the Fourth All-Union Conference on Thermomechanical Working [in Russian], Vol. 8, ITMO AN BelSSR, Minsk (1972), p. 541.
3. L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media [in Russian], GITTL, Moscow (1954), p. 245,
4. B. F. Jones, J. Math. Mech., 11, No. 6, 907 (1962).
5. J. Douglas and B. F. Jones, J. Math. Mech., 11, No. 6, 914 (1962).